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INSTANTON CALCULUS AND SUSY GAUGE THEORIES ON ALE MANIFOLDS

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ABSTRACT

We study instanton effects along the Coulomb branch of an $N = 2$ supersymmetric Yang–Mills theory with gauge group $SU(2)$ on Asymptotically Locally Euclidean (ALE) spaces. We focus our attention on an Eguchi–Hanson gravitational background and on gauge field configurations of lowest Chern class.

1 Introduction

Globally supersymmetric Yang–Mills (SYM) theories on four–manifolds [1] provide a natural framework in which to study non–perturbative effects. The existence of non–renormalization theorems [2, 3] allows one to exactly compute physical quantities like superpotentials in $N = 1$ theories [4]. Moreover, in $N = 2$ SYM theories, holomorphy requirements on the prepotential [5] are the crucial ingredients needed to determine the quantum moduli space and the Wilsonian effective action [6].

Instantons [7, 8] are among the most interesting non–perturbative field configurations. In particular, they proved to be a fundamental tool for checking, from first principles and quantitatively, the exactness of the solutions proposed by Seiberg and Witten in $N = 2$ SYM theory and supersymmetric QCD [9]. Furthermore, in some theories with matter in chiral representations, they are known to trigger dynamical SUSY breaking [10].

It is possible to perform instanton calculations in different phases of supersymmetric field theories. If the scalar fields of the theory have zero vacuum expectation value, instantons are exact saddle points of the action functional around which to perform semiclassical approximations. If the scalar fields have non–zero vacuum expectation values, instantons are just approximate solutions of the equations of motion. On the other hand, when the vacuum expectation values are much larger than the renormalization group invariant scale of the theory, it is possible to perform reliable instanton calculations in a weak–coupling regime ("constrained instanton" method [8, 11]).

In the following we will focus our attention on $N = 2$ SYM theories with gauge group $SU(2)$. In this case, when the complex scalar field has a non–zero vacuum expectation value, the gauge group is spontaneously broken down to $U(1)$. One can then study the dynamics of the low–energy theory which is obtained after integrating out the high–frequency modes. The first motivation of our computational approach consists in studying the non–perturbative dynamics of $N = 2$ effective theories on curved backgrounds via instanton calculus. To this end, the correct choice is to give non–zero vacuum expectation values to the scalar fields. One could thus infer the instanton corrections to the $N = 2$ holomorphic prepotential, which encodes the low–energy dynamics.

Among all possible four–manifolds a special class is represented by manifolds which have a self–dual Riemann tensor and which are known as “Asymptotically Locally Euclidean gravitational instantons”, since they are solutions of Einstein’s equations with vanishing gravitational action. These manifolds have played a key rôle in the study of Euclidean quantum gravity (for a review see [12]). Indeed, similarly to gauge instantons, they induce calculable non–perturbative effects which may cause a dynamical breaking of supersymmetry[1]. Among ALE gravitational instantons, the simplest and the most investigated one is the Eguchi–Hanson solution [13]. The formation of fermionic condensates in this background has been studied both in a pure supergravity [14, 15] and in an effective string theory context [16]. In particular, in [14] the gravitino field–strength condensate was explicitly computed and found to be finite and position–independent, possibly responsible for local supersymmetry breaking. Moreover, from a stringy point of view, ALE manifolds represent absolute minima of the gravitational part of the action which is obtained as a low–energy limit of the heterotic (and type I) string. Our second motivation is that in this context one could perform interesting string–inspired calculations, and explore a possible supersymmetry breaking (the underlying string theory acting as a regulator for the non–renormalizable supergravity theory)[2]. Gauge instantons of $c_2 = 1/2$ and 1 are solutions of the string equations of motion to lowest order in the σ–model coupling constant α'. The case $c_2 = 3/2$, instead, corresponds to the identification of the gauge connection with the spin connection [16] (“standard embedding”) and the solution is conjectured not to get perturbative corrections in α'. As a first step towards the case $c_2 = 3/2$ (which presents formidable computational difficulties) we start our investigation by studying correlation functions in the topological sector $c_2 = 1/2$, which could provide us with a useful roadmap for further progress along that direction.

The plan of the paper is as follows. In section 2 we briefly review $N = 2$ SYM theories on the Eguchi–Hanson manifold, mainly to fix our conventions. In section 3

[1]Generally, on curved manifolds SUSY is not globally realized. However, as pointed out in [1], in $N = 2$ SYM theories there exists a conserved scalar supercharge which can be interpreted as the BRST generator of the topological symmetry of the twisted version of the theory. In particular, this ensures that certain correlators of local operators are position–independent.

[2]A study of non–perturbative effects in global SYM theories in the Eguchi–Hanson background has been performed in the absence of vacuum expectation values for the scalar fields in [17].

we examine the (gauge) instanton configuration of Chern class $c_2 = 1/2$, around which we will expand the generating functional of Green's functions. On curved backgrounds, one must pay particular attention to the treatment of the collective coordinates which describe the instanton orientation in color space. We carefully discuss this issue and show how to correctly perform the integration over the related moduli in order to restore all the unbroken symmetries of the model. In this section we also collect the bosonic and fermionic zero–mode norms, and compute the classical Higgs and Yukawa actions. Section 4 is devoted to the (semiclassical) evaluation of some instanton–dominated correlators of the microscopic theory. In the final section we draw some conclusions and discuss some further issue under investigation.

2 Description of the model

We intend to study instanton–dominated correlators in $N = 2$ globally supersymmetric theories in the Eguchi–Hanson background.

The Eguchi–Hanson metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(\frac{r}{u}\right)^2 dr^2 + r^2(\sigma_x^2 + \sigma_y^2) + u^2 \sigma_z^2 \quad , \tag{2.1}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the left–invariant forms on $SU(2)$ and $u = r\sqrt{1-(a/r)^4}$. The metric (2.1) has a bolt singularity at $r = a$ which can be removed by changing to the radial variable u and by identifying antipodal points. Thus the boundary is $S^3/\mathbb{Z}_2$; moreover, the manifold is not invariant under the action of the Poincaré group, but admits an isometry group which is $SU(2)_R \otimes U(1)_L$. In particular, and this will be crucial in the following, the manifold is not translationally invariant since antipodal points are identified.

In order to study non–perturbative contributions to Green's functions we need to know the form of the gauge instantons[3]. On flat space, this can be achieved through the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction [19, 20], which naturally provides us with an algorithm which determines the most general self–dual instanton connection. Its extension to the case of ALE spaces was found by Kronheimer and Nakajima [21]. It was then translated in a more physical language in [22, 23], where

[3]Or, at least, a parametrization which describes the full moduli space of the instanton solution (see [18]).

the explicit expressions of the self–dual connection on the minimal instanton bundle $\mathcal{E}$ (with second Chern class $c_2(\mathcal{E}) = 1/2$) and of the bosonic and fermionic zero–modes were derived and implicit formulas for the cases $c_2 = 1$ and $3/2$ were given.

The $N = 2$ Super Yang–Mills action is $S_{\rm SYM} = \int d^4x\sqrt{g}\; L_{\rm SYM}$, where

$$\begin{aligned} L_{\rm SYM} &= \frac{2}{g^2}{\rm Tr}\left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\lambda}_A \not{D}\lambda^A + (D^\mu\phi)^\dagger(D_\mu\phi) + \right. \\ &\left. + \frac{1}{2}[\phi,\phi^\dagger]^2 + \left(\frac{1}{\sqrt{2}}[\phi^\dagger,\lambda_A]\lambda_B\varepsilon^{AB} + {\rm h.c.}\right)\right] \quad , \end{aligned} \tag{2.2}$$

$A, B = 1, 2$ are supersymmetry indices, and $\varepsilon_{12} = -\varepsilon_{21} = 1$[4].

The condition for the vacuum state of the theory to be $N = 2$ supersymmetric is that the potential $V(\phi, \phi^\dagger)$ vanishes, that is

$$[\phi, \phi^\dagger] = 0 \; . \tag{2.3}$$

This means that the solution of (2.3) is a normal operator, which can therefore be diagonalized by an $SU(2)$ color rotation Ω. Then, the classical[5] (supersymmetric) vacuum configuration for the Higgs field can be written as

$$\langle\phi^a\rangle = \Omega^a{}_b(v\delta^{b3}) \quad , \tag{2.4}$$

where $v \in \mathbb{C}$ and $a, b = 1, 2, 3$. One can then expand (2.4) in the basis

$$\phi_0^{a(b)} = v\delta^{ab} \quad . \tag{2.5}$$

When $v \neq 0$ the gauge symmetry is spontaneously broken to $U(1)$, and we are left with an $N = 2$ SUSY abelian low–energy theory.

[4]We choose the generators in the fundamental representation to be $T^a = \tau^a/2$, τ^a being the Pauli matrices.

[5]Actually neither perturbative nor non–perturbative quantum corrections can lift the vacuum degeneracy [24, 25].

3 The instanton configuration and the choice of collective coordinates

The gauge instanton solution for $c_2(\mathcal{E}) = 1/2$ is given by acting with a global color rotation R on the basic instanton configuration [22]

$$A = A_\mu dx^\mu = i \begin{pmatrix} f(r)\sigma_z & g(r)\sigma_- \\ g(r)\sigma_+ & -f(r)\sigma_z \end{pmatrix} \quad , \tag{3.1}$$

where $\sigma_\pm = \sigma_x \pm i\sigma_y$, and

$$f(r) = \frac{t^2 r^2 + a^4}{r^2(r^2+t^2)} \quad , \qquad g(r) = \frac{\sqrt{t^4 - a^4}}{r^2 + t^2} \quad . \tag{3.2}$$

The matrix R contains the collective coordinates related to global color rotations. When $a = 0$ this configuration becomes the 't Hooft instanton in the so–called "singular gauge", centered around the origin.

To compute instanton–dominated Green's functions one has to know the explicit form of the bosonic and fermionic zero–modes. In the $N = 2$ Eguchi–Hanson background there exist four gaugino zero–modes which are related to the fact that the instanton solution explicitly breaks the superconformal symmetry. They can be written as

$$\lambda^{aA}_{\alpha(0)} = \sigma^\mu_{\alpha\dot\alpha} (D_\mu \theta)^a \bar\varepsilon^{A\dot\alpha} \quad , \tag{3.3}$$

where $\bar\varepsilon^{A\dot\alpha}$ are the two covariantly constant spinors on the Eguchi–Hanson background and θ_a (with $a = 1, 2, 3$) are the bounded solutions of the scalar Laplace equations

$$[D^2(A)]_a{}^b \theta_b = 0 \quad . \tag{3.4}$$

These equations can be recast (with a radial Ansatz) in the form

$$\left[\frac{1}{r^3}\frac{\partial}{\partial r}(ru^2)\frac{\partial}{\partial r} - 4\left(\frac{g^2}{r^2} + \frac{f^2}{u^2}\right)\right]\theta_{1,2} = 0 \quad ,$$

$$\left[\frac{1}{r^3}\frac{\partial}{\partial r}(ru^2)\frac{\partial}{\partial r} - \frac{8g^2}{r^2}\right]\theta_3 = 0 \quad . \tag{3.5}$$

The solutions to (3.5) with boundary conditions $\lim_{r\to\infty}\theta_a(r) = 1$ are given by

$$\theta_1 = \theta_2 = \frac{\sqrt{r^4 - a^4}}{r^2 + t^2} \quad , \qquad \theta_3 = \frac{t^2 r^2 + a^4}{t^2(r^2+t^2)} \quad . \tag{3.6}$$

The most general form of the gaugino zero–modes can be re–written in terms of two constant spinors η_α^A as the global color rotation R acting on the configuration (3.3), that is

$$[\lambda^{aA}_{\alpha(0)}]_R = R^a{}_b(f^b\sigma^{b\,\beta}_\alpha\eta^A_\beta) \quad , \tag{3.7}$$

where

$$\begin{aligned} f^1 &= f^2 = \frac{2(t^2r^2+a^4)}{r(r^2+t^2)^2} \quad , \\ f^3 &= \frac{2\sqrt{(t^4-a^4)(r^4-a^4)}}{r(r^2+t^2)^2} \quad . \end{aligned} \tag{3.8}$$

The norm of the gaugino zero–modes is

$$||\lambda||^2 = \int d^4x\sqrt{g}\,(\lambda^a_\alpha)^*(\bar\sigma^0)^{\dot\alpha\beta}\lambda^a_\beta = \sqrt{2}\pi t \quad . \tag{3.9}$$

The four zero–modes of the gauge field are related to the global symmetries broken by the instanton background, *i.e.* dilatations and $SU(2)$ rotations. The zero–mode related to dilatations is

$$\delta_0 A = \frac{\partial A}{\partial t} = \frac{it}{\sqrt{t^4-a^4}}\begin{pmatrix} f^3(r)u\sigma_z & f^1(r)r\sigma_- \\ f^1(r)r\sigma_+ & -f^3(r)u\sigma_z \end{pmatrix} \quad , \tag{3.10}$$

and satisfies the usual background gauge condition $D^\mu\delta_0A_\mu = 0$. The three zero–modes related to global $SU(2)$ color rotations are not transverse, but they can be made such by adding a local gauge transformation [22]. The resulting transverse zero–modes are then

$$\delta_a A^b_\mu = (D_\mu\theta_a)^b \quad . \tag{3.11}$$

The four bosonic zero–modes are now orthogonal and their norms are

$$\begin{aligned} ||\delta_0 A||^2 &= \frac{8\pi^2t^4}{t^4-a^4} \quad , \\ ||\delta_1 A||^2 &= ||\delta_2 A||^2 = 8\pi^2t^2 \quad , \\ ||\delta_3 A||^2 &= \frac{8\pi^2(t^4-a^4)}{t^2} \quad . \end{aligned} \tag{3.12}$$

Here a subtle point arise: collective coordinates are associated to unbroken symmetries of the theory. Namely, the related transformations leave the chosen vacuum configuration

$$\langle A_\mu\rangle = 0 \quad , \qquad \langle\phi^a\rangle = \Omega^a{}_b(v\delta^{b3}) \tag{3.13}$$

unchanged. When the gauge symmetry is spontaneously broken, we may expect that the corresponding collective coordinates should not be taken into account. This question has been carefully studied in [26]. It was pointed out there that we can act on the family of vacuum configurations (2.5) in two different ways. Rotations acting from the left obviously correspond to $SU(2)$ global color transformations. Rotations acting from the right are however also possible (they are called “flavor” rotations in [26]). The crucial observation is that the basis (2.5) is left invariant when the two rotations are realized by the same matrix R. In other words,

$$R^a{}_d\, \phi_0^{d(c)} (R^T)_c{}^b = \phi_0^{a(b)} \quad . \tag{3.14}$$

This $SU(2)$ flavor symmetry exists only in the space of classical solutions for the Higgs field, but not at the Lagrangian level. In particular it acts trivially on the gauge sector and it does not affect the structure of the gauge zero–modes. In flat space the previous observations have no effect; in our case, however, we are led to consider the solution of the scalar Laplace equation (3.4) with boundary conditions dictated by (2.4). A basis of three independent solutions is given by

$$\phi^{a(b)}(x) = \phi_0^{a(b)} \theta^a(x) = v\Big[\theta^1 \delta^{ab} + (\theta^3 - \theta^1)\delta^{a3}\delta^{b3}\Big] \quad . \tag{3.15}$$

The point is that, in order to ensure a correct vacuum alignment, one is forced to act on $\phi^{a(b)}$ with the rotations mentioned above, that is

$$\phi^{a(b)} \longrightarrow R^a{}_d\, \phi^{d(c)} (R^T)_c{}^b \equiv \phi_R^{a(b)} \quad . \tag{3.16}$$

If we choose the boundary condition

$$\langle \phi^a \rangle = v\delta^{a3} \tag{3.17}$$

for the Higgs field, the correct solution for finite x is[6]

$$\phi^a_{\rm cl}(x) = \phi_R^{a(3)}(x) \quad . \tag{3.18}$$

The non–triviality of (3.18) resides in the different expressions of the θ^a which are a consequence of the isometry group of the Eguchi–Hanson manifold. On the other hand,

[6] Different choices of the matrix Ω in (3.13) give physically equivalent theories; so we put $\Omega = 1\!\!1$.

in flat space one has $\theta^a_{\rm flat} = x^2/(x^2+\rho^2)$, $a = 1,2,3$ (in the singular gauge) and the corresponding expression (3.18) for $\phi^a_{\rm cl}$ does not contain the matrix R anymore.
The expression (3.18) could also be obtained in a more direct way[7]. The scalar field configuration $\phi^a_{\rm cl}(x)$ can actually be found by simply requiring that it satisfies the scalar Laplace equation in the background of the most general (*i.e.* gauge–rotated) instanton configuration, RA, that is

$$[D^2(RA)]_a{}^b \phi^b_{\rm cl} = 0 \quad , \tag{3.19}$$

with the boundary condition (3.17). Since (3.19) is equivalent to

$$[D^2(A)]_a{}^b (R^T\phi_{\rm cl})^b = 0 \quad , \tag{3.20}$$

we can immediately convince ourselves that (3.18) satisfies (3.19) and (3.17).
The matrix R can be written in terms of three Euler angles θ, φ, ψ. As explained before, these angles are in fact the global color collective coordinates related to $SU(2)/\mathbb{Z}_2$. This is the way these instanton moduli come into play in this context.

With these elements we can now calculate the contribution of the Higgs field configuration to the classical action, which reads

$$S_{\rm cl} = \frac{1}{g^2}\int d^4x\sqrt{g}\left[(D^\mu\phi^\dagger_{\rm cl})^a(D_\mu\phi_{\rm cl})^a\right] = \frac{2\pi^2t^2|v|^2}{g^2}\left(1-\frac{a^4}{t^4}\cos^2\theta\right) \quad . \tag{3.21}$$

Note in (3.21) the explicit dependence on the gauge orientations.

Let us now calculate the Yukawa action $S_{\rm Y}$ written with the complete expansion of the fermionic fields replaced by their projection over the zero–mode subspace. According to the index theorem for the Dirac operator in the background of a self–dual gauge field configuration, we have only zero–modes of one chirality, so $S_{\rm Y}$ reduces to

$$S_{\rm Y}\left[\phi,\phi^\dagger,\lambda^{(0)},\bar\lambda=0\right] = \frac{\sqrt{2}}{g^2}\int d^4x\sqrt{g}\;\varepsilon_{abc}(\phi^\dagger_{\rm cl})^a(\lambda^b_{(0)}\psi^c_{(0)}) \quad , \tag{3.22}$$

where, for the sake of clarity, we adopted different symbols for the two gauginos, $\lambda = \lambda_1$ and $\psi = \lambda_2$. Inserting (3.7) and (3.18) in (3.22), we get

$$S_Y = -\frac{2i\sqrt{2}}{g^2}\int d^4x\sqrt{g}\left[\theta_1 f_1 f_3(\delta_{b1}w_1^* + \delta_{b2}w_2^*) + \theta_3(f_1)^2\delta_{b3}w_3^*\right](\eta_{[\lambda]}\sigma^b\eta_{[\psi]}) \quad , \tag{3.23}$$

[7]We thank Gian Carlo Rossi for discussions on this point.

where $\eta_{[\lambda]} = \eta_1$ and $\eta_{[\psi]} = \eta_2$ and we have defined

$$
\begin{aligned}
w_1 &= v \sin\theta \sin\psi \quad , \\
w_2 &= v \sin\theta \cos\psi \quad , \\
w_3 &= v \cos\theta \quad .
\end{aligned}
\tag{3.24}
$$

After a straightforward integration we finally obtain

$$
S_Y = \eta_{[\lambda]}\sigma^b M^{b*}\eta_{[\psi]} \equiv \frac{-i\sqrt{2}\pi^2}{g^2}(\eta_{[\lambda]}\sigma^b\eta_{[\psi]})\left[(\delta_{b1}w_1^* + \delta_{b2}w_2^*)\sqrt{t^4 - a^4} + \delta_{b3}w_3^*\frac{a^4 + t^4}{t^2}\right] \quad . \tag{3.25}
$$

4 Computation of instanton–dominated Green's functions

Let us now compute the simplest non–zero correlator in the Eguchi–Hanson background. Since the base manifold is not translationally invariant, there are no supersymmetric gaugino zero–modes, unlike the case of flat space. Moreover, as there is a non–zero vacuum expectation value for the scalar field, the superconformal gaugino zero–modes are lifted. The integration over the fermionic collective coordinates is thus entirely saturated by the Yukawa action, and the simplest non–zero correlator is $\langle 1\!\!1 \rangle$, *i.e.* the partition function itself. The integration over the bosonic zero–modes is then replaced by an integration over the moduli of the instanton. The corresponding Jacobian is [17]

$$
J = \prod_{I=0}^{3} \frac{||\delta_I A||}{\sqrt{2\pi}} = \frac{64\pi^4 t^3}{(\sqrt{2\pi})^4} \quad , \tag{4.1}
$$

which does not depend on a. This is not an unexpected result, since the metric on the minimal instanton moduli space coincides with the Eguchi–Hanson metric [22].

The evaluation of the Green's function $\langle 1\!\!1 \rangle$ in the semiclassical approximation yields, after integrating over non–zero mode fluctuations,

$$
\begin{aligned}
\langle 1\!\!1 \rangle &= e^{-\frac{8\pi^2}{g^2}\frac{1}{2}}\mu^2 \int_a^\infty dt \int_{SU(2)/Z_2} d^3\Sigma \frac{64\pi^4 t^3}{(\sqrt{2\pi})^4}\left(\frac{1}{\sqrt{2}\pi t}\right)^4 \det M^* \times \\
&\times \ \exp\left[-\frac{2\pi^2 t^2 |v|^2}{g^2}\left(1 - \frac{a^4}{t^4}\cos^2\theta\right)\right] \quad ,
\end{aligned}
\tag{4.2}
$$

where $d^3\Sigma = \frac{1}{8}\sin\theta d\theta d\varphi d\psi$, $M^* = \sigma^b M^{b*}$ and $\det M^*$ comes from integrating $\exp(-S_{\rm Y})$. Furthermore, $\mu^{4-\frac{1}{2}(2+2)} e^{-\frac{8\pi^2}{2g^2}} = \Lambda^2$, where Λ is the $N=2$ SYM renormalization group invariant scale with gauge group $SU(2)$. The scale μ comes from the Pauli–Villars regularization of the determinants, and the exponent is $b_1 c_2(\mathcal{E}) = n_B - n_F/2$, where n_B, n_F are the number of bosonic and fermionic zero–modes and b_1 is the first coefficient of the β–function of the theory. Finally, the factor $\exp(-8\pi^2/2g^2)$ comes from the instanton action. Writing the determinant explicitly we obtain:

$$\langle 1\!\!1 \rangle = -\frac{4\pi^4 (v^*)^2 \Lambda^2}{g^4} I(a) \quad , \tag{4.3}$$

where

$$I(a) = \int_a^\infty \frac{dt}{t} \int_{-1}^1 dy\; e^{-\frac{2\pi^2 |v|^2 t^2}{g^2}(1-y^2\frac{a^4}{t^4})} \left[(1-y^2)(t^4-a^4) + y^2 \left(\frac{a^4+t^4}{t^2} \right)^2 \right] \quad , \tag{4.4}$$

and $y = \cos\theta$. In the limit $a \to 0$ in which the Eguchi–Hanson manifold approaches the orbifold $\mathbb{R}^4/\mathbb{Z}_2$, the integral becomes

$$I(a \to 0) = \frac{g^4}{4\pi^4 |v|^4} \quad , \tag{4.5}$$

so that

$$\langle 1\!\!1 \rangle_{a\to 0} = -\frac{\Lambda^2}{v^2} \quad . \tag{4.6}$$

In the general case $a \neq 0$, after some algebraic manipulations, one gets

$$\begin{aligned} I(a) &= \sum_{n=0}^\infty \frac{a^4 x^{2n-2}}{n!(2n+1)} \Gamma(2-n,x) + \sum_{n=0}^\infty \frac{4n a^4 x^{2n}}{n!(2n+1)(2n+3)} \Gamma(-n,x) + \\ &+ \sum_{n=0}^\infty \frac{a^4 x^{2n+2}}{n!(2n+3)} \Gamma(-2-n,x) \quad , \end{aligned} \tag{4.7}$$

where $x = 2\pi^2 a^2 |v|^2/g^2$, and $\Gamma(n,x)$ is Euler's incomplete gamma function. In the limit $a \to 0$ we recover (4.5).

As one could expect, the correlator explicitly depends on the Eguchi–Hanson parameter a. Indeed, the dependence on Λ is completely fixed by zero–modes counting to be $\Lambda^{n_B - \frac{1}{2} n_F}$ (in the present case $n_B = n_F = 4$). In the case $v \neq 0$, however, one can form the adimensional quantity $a|v|$. Therefore, Ward–Takahashi identities and dimensional analysis do not completely fix the correlator dependence on v and a. On the other hand, when $v = 0$, the same argument dictates the independence of the correlators on the inverse mass scale a.

We now extend our analysis to the Green's function $\langle \mathrm{Tr}\phi^2 \rangle$. The quantum fluctuations of the complex scalar field are replaced, after functional integration, by ϕ_{inh}, where

$$\phi^a_{\mathrm{inh}} = \sqrt{2}\, \varepsilon^{bdc} [(D^2)^{-1}]^{ab} (\lambda^d_{(0)} \psi^c_{(0)}) \quad . \tag{4.8}$$

So, in this case we need one more ingredient, that is the solution of the equations of motion for the scalar field in the gaugino zero–modes background

$$(D^2 \phi_{\mathrm{inh}})^a = \sqrt{2}\, \varepsilon^a{}_{bc} (\lambda^b_{(0)} \psi^c_{(0)}) \quad . \tag{4.9}$$

The solution of (4.9) is given by

$$\phi^b_{\mathrm{inh}} = -2i\sqrt{2} h^b (\eta_{[\lambda]} \sigma^b \eta_{[\psi]}) \equiv m^b (\eta_{[\lambda]} \sigma^b \eta_{[\psi]}) \quad , \tag{4.10}$$

where the index b is not summed over and the functions h^b are

$$\begin{aligned} h^1 &= h^2 = -\frac{\sqrt{(t^4 - a^4)(r^4 - a^4)}}{4(r^2+t^2)^2} \quad , \\ h^3 &= -\frac{(a^4+t^4) r^2 + 2a^4 t^2}{4t^2 (r^2+t^2)^2} \quad . \end{aligned} \tag{4.11}$$

Let us now start our calculation. Unlike the case $v = 0$ there is more than one contribution from the insertion of the operator $\mathrm{Tr}\phi^2$ in the functional integral. In fact, all the remaining superconformal gaugino zero–modes are lifted, so that also the partition function gets a contribution from the instanton background. If we separate the classical and the quantum contribution from ϕ^b we obtain that the integration over the fermionic collective coordinates becomes

$$\begin{aligned} &\int d^2\eta_{[\lambda]} d^2\eta_{[\psi]}\; e^{-S_{\mathrm{Y}}} (\phi^b_{\mathrm{cl}} + \phi^b_{\mathrm{inh}})(\phi^b_{\mathrm{cl}} + \phi^b_{\mathrm{inh}}) = \\ &= (\det M^*)(\phi^b_{\mathrm{cl}} \phi^b_{\mathrm{cl}}) + 2 m^b m^b - 4(\phi^b_{\mathrm{cl}} m^b M^{b*}) \quad , \end{aligned} \tag{4.12}$$

where m^b is defined in (4.10). Due to the extreme complexity of the integrals, we limited ourselves to study the $a \to 0$ limit, in which (4.12) becomes

$$\begin{aligned} \widehat{I}(r,t) &= \int d^2\eta_{[\lambda]} d^2\eta_{[\psi]}\; e^{-S_{\mathrm{Y}}} (\phi^b_{\mathrm{cl}} + \phi^b_{\mathrm{inh}})(\phi^b_{\mathrm{cl}} + \phi^b_{\mathrm{inh}}) = \\ &= -\frac{r^4}{(r^2+t^2)^2} \left(\frac{2\pi^4 |v|^4}{g^4} + \frac{4\pi^2 |v|^2}{g^2 (r^2+t^2)} + \frac{3}{(r^2+t^2)^2} \right) \quad . \end{aligned} \tag{4.13}$$

The correlator has thus the form

$$\langle \mathrm{Tr}\phi^2 \rangle_{a\to 0} = \frac{1}{2}\Lambda^2 \int_0^\infty dt \int_{SU(2)/Z_2} d^3\Sigma \; \frac{64\pi^4 t^3}{(\sqrt{2\pi})^4} \frac{1}{(\sqrt{2}\pi t)^4} e^{-\frac{2\pi^2 t^2 |v|^2}{g^2}} \widehat{I}(r,t) = \frac{\Lambda^2}{2} \quad , \quad (4.14)$$

and is position–independent, as required by supersymmetry. Dimensional analysis and zero–modes counting completely constrains the correlator in (4.14) to be independent of the mass scale v.

5 Discussion

In this paper we have computed some correlators in an $SU(2)$, $N = 2$ SYM on ALE backgrounds. The choice of a non–zero vacuum expectation value for the scalar field introduces a new scale v. On dimensional grounds, when $v = 0$, instanton–dominated correlators obviously do not depend on a [17]. However, the study of instanton effects in the low–energy theory requires setting $v \neq 0$ from the beginning. The Green's functions we have studied show an explicit dependence on the adimensional quantity $a|v|$. Generally speaking, we expect all correlators to depend on this quantity. This means that an extension of our calculations to supergravity, where a becomes itself a collective coordinate to integrate over, is not straightforward. Indeed, we cannot appeal to an explicit factorization between the gauge and the gravitational sectors and thus exploit the results found in [14]. We want to remark that, as in flat space, the same correlation function has different values when calculated in different phases of the theory (that is, in the presence or in the absence of a vacuum expectation value for the scalar field). This point deserves further investigations.

In [27], microscopic Green's functions were directly related to the $N = 2$ abelian holomorphic prepotential (in the flat space case). It would be interesting to check (*e.g.* with instanton techniques) if this approach can be extended to our context. We also plan to extend our analysis to $N = 4$ SYM theories and to the case $c_2 = 1$ in a future publication.

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